Positivity and lower bounds for the density of Wiener functionals

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Abstract. We consider a functional on the Wiener space which is smooth and not degenerated in Malliavin sense and we give a criterion of strict positivity of the density. We also give lower bounds for the density. These results are based on the representation of the density by means of the Riesz transform introduced in Malliavin and Thalmaier in [14] and on the estimates of the Riesz transform given in Bally and Caramellino [3].

Keywords: Riesz transform, Malliavin calculus, strict positivity and lower bounds of the density.

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1 Introduction

The aim of this paper is to study the strict positivity and lower bounds for the density of a functional on the Wiener space. Although the two problems are related each other, the hypothesis under which such results may be obtained are different. Just to make clear what we expect to be these hypothesis, consider the example of a d dimensional diffusion process X_t solution of $dX_t = \sum_{j=1}^m \sigma_j(X_t) \circ dW_t^j + b(X_t)dt$ where $\circ dW_t^j$ denotes the Stratonovich integral. The skeleton associated to this diffusion process is the solution $x_t(\phi)$ of the equation $dx_t(\phi) = \sum_{j=1}^m \sigma_j(x_t(\phi))\phi_t^j dt + b(x_t(\phi))dt$, for a square integrable ϕ. The celebrated support theorem of Stroock and Varadhan guarantees that the support of the law of X_t is the closure of the set of points x which are attainable by a skeleton, that is $x = x_t(\phi)$ for some control $\phi \in L^2([0,T])$. Suppose now that the law of X_t has a continuous density p_{X_t} with

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respect to the Lebesgue measure. Then we want to have a criterion for $p_{X_t}(x) > 0$. And we prove that this is true if x is attainable, that is $x = x_t(\phi)$ for some ϕ, and a suitable non degeneracy assumption holds in x. The second problem is to give a lower bound for $p_{X_t}(x)$ and this can be achieved if a non degeneracy condition holds all along the curve $x_t(\phi)$ which arrives in x. Roughly speaking the idea is the following: one takes a tube around the curve $x_t(\phi)$ which arrives in x. If one has a non degeneracy condition all along the curve then one may give a lower bound for the probability to remain in the tube up to $t-\delta$ for a small $\delta > 0$ and then one employs an argument based on Malliavin calculus in order to focus on the point x - essentially this means that one is able to give a precise estimate of the behavior of the diffusion in short time (between $t-\delta$ and t). And so we obtain the lower bound for $p_{X_t}(x)$. If one does not need a lower bound but only the strict positivity, the argument is the same but one does not need to estimate the probability to remain in the tube: using the support theorem one knows that this probability is strictly positive (but this is just qualitative, so one has no lower bound for it) and then one focuses on the point x using again the same argument concerning the behavior of the diffusion in short time. So one needs the non degeneracy condition in x only.

The two problems mentioned above have been intensively studied in the literature. Let us begin with the strict positivity. At the best of our knowledge the first probabilistic approach to this problem is due to Ben-Arous and Leandre [7] who used Malliavin calculus in order to give necessary and sufficient conditions in order to have $p_{X_t}(x) > 0$ for a diffusion process (as above). They proved that if Hörmander's condition holds then a necessary and sufficient condition in order to have $p_{X_t}(x) > 0$ is that x is attainable by a skeleton $x_t(\phi)$ such that $\psi \mapsto x_t(\psi)$ is a submersion in ϕ. The argument they used is based on the inverse function theorem and on a Girsanov transformation. All the papers which followed used in a way or in another their argument. First, Aida, Kusuoka and Stroock [1] gave a generalization of this criterion in an abstract framework which still permits to exhibit a notion of skeleton. Then Hirsch and Song [10] gave a variant of such a criterion for a general functional on the Wiener space using capacities and finally Leandre [13] obtained such a criterion for diffusion processes on manifolds. Notice that once we have a criterion of the above type there is still a non trivial problem to be solved: one has to exhibit the skeleton which verifies the submersion property. So number of authors dealt with concrete examples in which they are able to use in a more or less direct way the argument of Ben-Arous and Leandre: Bally and Pardoux [6] deal with parabolic stochastic heat equations, Millet and Sanz-Solé [16] work with hyperbolic stochastic partial differential equations, Fournier [9] deals with a jump type equation, Dalang and D. Nualart [8] use such positivity results for building a potential theory for SPDE's and E. Nualart [18] has recently proved results in this direction again for solutions of SPDE's.

Concerning lower bounds for the density, a first result was found by Kusuoka and Stroock [12] for diffusion processes which verify a strong uniform Hörmander condition. Afterwards Kohatsu-Higa [11] obtained lower bounds for general functionals

on the Wiener space under a uniform ellipticity condition and Bally [2] proved results under local ellipticity conditions. Recently, Gaussian type lower and upper bounds are studied in E. Nualart [19] for the nonlinear stochastic heat equation.
This paper gives a contribution in this framework. In fact, we study the strict positivity and lower bounds for the density of a general functional on the Wiener space (Theorem 7) as a consequence of a result (Lemma 5) which gives the behavior of a small perturbation of a Gaussian random variable - it corresponds to the study of a diffusion process in short time (between $t-\delta$ and t) and is of interest in itself. Here, we use the representation of the density in terms of the Riesz transform introduced by Malliavin and Thalmaier [14]. As studied in Bally and Caramellino [3], this allows one to ask for less regularity for the functional at hand and mainly, the constants involved in the lower bounds depend on the Malliavin-Sobolev norms up to order 4 does not matter the dimension d of the functional (while in [2] they depend on the Malliavin-Sobolev norms up to order $d+3$). And this is a concrete gain in writing the estimates. We then apply our results to some examples in which a support theorem is available, and we can see in practice how non degeneracy in attainable points or along the whole skeleton path effect the results on the strict positivity or on the lower bounds for the density respectively. As a consequence, e.g. in relation to the result of Ben-Arous and Leandre, we actually obtain a new criterion which represents only a sufficient condition for the strict positivity of the density but is rather explicit and easy to check.
In our examples, we first deal with an Ito process X_t defined as a component of a diffusion process, that is

$$\begin{aligned} X_t =& x_0 + \sum_{j=1}^m \int_0^t \sigma_j(X_t, Y_t) dW_t^j + \int_0^t b(X_t, Y_t) dt \\ Y_t =& y_0 + \sum_{j=1}^m \int_0^t \alpha_j(X_t, Y_t) dW_t^j + \int_0^t \beta(X_t, Y_t) dt. \end{aligned}$$

Notice that for diffusion processes, we get an example which is essentially the same treated in Ben Arous and Leandre [7] and in Aida, Kusouka and Stroock [1]. Let $(x(\phi), y(\phi))$ denote the skeleton associated to the diffusion pair (X, Y) and let $x = x_t(\phi)$ for some suitable control ϕ. Then, whenever a local density p_{X_t} of X_t exists in x, we prove that if $\sigma\sigma^*(x, y_t(\phi)) > 0$ then $p_{X_t}(x) > 0$. And moreover, if $\inf_{s\leq t} \inf_y \sigma\sigma^*\big(x_s(\phi), y\big) \geq \lambda_* > 0$ and $x_s(\phi)$ belongs to a suitable class of paths (see Theorem 8 for details), then a local density p_{X_t} exists in x and a lower bound for $p_{X_t}(x)$ can be written in terms of the lower estimates for the probability that Ito processes remain near a path proved in Bally, Fernández and Meda in [5].
As a second example, in Section 4.2 we treat the two dimensional diffusion process

$$dX_t^1 = \sigma_1(X_t) dW_t + b_1(X_t) dt, \quad dX_t^2 = b_2(X_t) dt$$

which is degenerated in any point $x \in \mathbb{R}^2$. We assume that x is attainable by a skeleton $x_t(\phi)$ and that $|\sigma_1(x)| > 0$ and $|\partial_1 b_2(x)| > 0$ - which amounts to say that the

weak Hörmander condition holds in the point x. We prove that under this hypothesis $p_{X_t}(x) > 0$. For this example Bally and Kohatsu-Higa [4] have already given a lower bound for the density under the stronger hypothesis that $\inf_{s\leq t}|\sigma(x_s(\phi))| > 0$ and $\inf_{s\leq t}|\partial_1 b_2(x_s(\phi))| > 0$. So the same non degeneracy condition holds but along the whole curve $x_s(\phi), 0 \leq s \leq t$. Notice that we use a skeleton $x_s(\phi)$ which arrives in x but we do not ask the immersion property (according the result of Ben-Arous and Leandre it follows that a skeleton which verifies the immersion property exists also, but we do not know how to produce it directly and we do not need it). And it seems clear to us that our criterion may be used for SPDE's as well and would simplify the proofs given in the already mentioned papers. We stress again that a necessary prerequisite in order to have a chance to apply our criterion seems to be the existence of a support theorem.
The paper is organized as follows. In Section 2 we recall some results from Bally and Caramellino [3] concerning the representation of the density by means of the Riesz transform. Section 3 refers to the results on the perturbation of a Gaussian random variable (see Section 3.1) and on the strict positivity and the lower bounds for the density of a general functional on the Wiener space (see Section 3.2). We finally discuss our examples in Section 4.

2 Localized Integration by Parts Formulas

We consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with an infinite dimensional Brownian motion $W = (W^n)_{n\in\mathbb{N}}$ and we use the Malliavin calculus in order to obtain integration by parts formulas. We refer to D. Nualart [17] for notation and basic results. We denote by $\mathbb{D}^{k,p}$ the space of the random variables which are k times differentiable in Malliavin sense in L^p and for a multi-index $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$ we denote by $D^\alpha F$ the Malliavin derivative of F corresponding to the multi-index α. So, $\mathbb{D}^{m,p}$ is the closure of the space of the simple functionals with respect to the Malliavin Sobolev norm

$$\|F\|_{m,p}^p = \|F\|_p^p + \sum_{k=1}^m \sum_{|\alpha|=k} \mathbb{E}\Big(\Big(\int_{[0,\infty)^k} \big|D^\alpha_{s_1,\dots,s_k} F\big|^2 ds_1 \dots ds_k\Big)^{p/2}\Big).$$

We will use the following notation: for $F = (F^1, \dots, F^d)$, $F^i \in \mathbb{D}^{m,p}$, we set

$$|D^{(m)}F|^2 = \sum_{|\alpha|=m} |D^\alpha F|^2 = \sum_{i=1}^d \sum_{|\alpha|=m} |D^\alpha F^i|^2.$$

Moreover, for $F = (F^1, \dots, F^d)$, $F^i \in \mathbb{D}^{1,2}$, we let σ_F denote the Malliavin covariance matrix associated to F :

$$\sigma_F^{i,j} = \langle DF^i, DF^j\rangle = \sum_{k=1}^\infty \int_0^\infty D_s^k F^i D_s^k F^j ds, \quad i,j = 1, \dots, d.$$

The non-degeneracy condition is given by

$$(\det \sigma_F)^{-1} \in \cap_{p\in\mathbb{N}} L^p. \tag{1}$$

Under (1), we denote by $\widehat{\sigma}_F$ the inverse matrix. We also denote by δ the divergence operator (Skorohod integral) and by L the OrnsteinUhlenbeck operator and we recall that if $F \in \cap_{p\in\mathbb{N}}\mathbb{D}^{2,p}$ then $F \in Dom(L)$. The following proposition gives the classical integration by parts formula from Malliavin calculus.

Proposition 1. *i) Let $F = (F^1, \dots, F^d)$ with $F^1, \dots, F^d \in \cap_{p\in\mathbb{N}}\mathbb{D}^{2,p}$ and $G \in \cap_{p\in\mathbb{N}}\mathbb{D}^{1,p}$. Assume that (1) holds. Then for every function $f \in C_b^1(\mathbb{R}^d)$ one has*

$$\mathbb{E}(\partial_i f(F)G) = -\mathbb{E}(f(F)H_i(F,G)) \quad \text{for every } i = 1,\dots d \tag{2}$$

where

$$H_i(F,G) = -\sum_{j=1}^{d} \delta(G\widehat{\sigma}_F^{ji} DF^j) = -\sum_{j=1}^{d} \Big(G\widehat{\sigma}_F^{ji} L(F^j) + \big\langle DF^j, D(\widehat{\sigma}_F^{ji} \times G)\big\rangle \Big) \in \bigcap_{p\in\mathbb{N}} L^p.$$

Moreover, for every $p \ge 1$ there exists universal non negative constants $C_{p,d}, q_{p,d}, \ell_{p,d}$ such that

$$\|H(F,G)\|_p \le C_{p,d} \times \|(\det \sigma_F)^{-1}\|_{q_{p,d}} \times \|G\|_{1,q_{p,d}} \times (1 + \|F\|_{2,q_{p,d}})^{\ell_{p,d}}.$$

ii) Suppose that $F^1, \dots, F^d \in \cap_{p\in\mathbb{N}}\mathbb{D}^{k+1,p}$ and $G \in \cap_{p\in\mathbb{N}}\mathbb{D}^{k,p}$ for some $k \in \mathbb{N}$. Then for every multi-index $\alpha = (\alpha_1, \dots, \alpha_k) \in \{1, \dots, d\}^k$ one has

$$\mathbb{E}(\partial_\alpha f(F)G) = \mathbb{E}(f(F)H_\alpha(F,G)) \quad \text{with} \quad H_\alpha(F,G) = H_{\alpha_k}(F, H_{(\alpha_1,\dots,\alpha_{k-1})}(F,G)) \tag{3}$$

and $H_\alpha(F;G) \in \cap_{p\in\mathbb{N}} L^p$.

Proposition 1 allows to get a representation formula of the density and the condition expectation in terms of the Riesz transform, that we briefly recall now.
We denote by Q_d the Poisson kernel, i.e. the fundamental solution of the equation $\Delta Q_d = \delta_0$ in $\mathbb{R}^d$. Q_d has the following explicit form:

$$Q_1(x) = \max(x,0), \quad Q_2(x) = a_2^{-1} \ln|x| \quad \text{and} \quad Q_d(x) = -a_d^{-1} |x|^{2-d}, d > 2 \tag{4}$$

where for $d \ge 2$, a_d is the area of the unit sphere in $\mathbb{R}^d$.
In what follows, we need some results that we resume in the next proposition.

Proposition 2. *Let $F = (F^1, \dots, F^d)$ and assume that (2) holds with $G = 1$ and some wights $H_i(F,1) \in \cap_{p\in\mathbb{N}} L^p$ for every $i = 1, \dots, d$.*
i) For any $p > d$ one has

$$\sup_{x\in\mathbb{R}^d} \sum_{i=1}^{d} \Big(\mathbb{E}(|\partial_i Q_d(F-x)|^{p/(p-1)})\Big)^{(p-1)/p} \le K_{d,p}\Big(1 + \sum_{i=1}^{d} \|H_i(F;1)\|_p\Big)^{k_{d,p}}$$

for some positive constants $k_{d,p}$ and $K_{d,p}$ depending on d and p only. Moreover, the law of F is absolutely continuous w.r.t. the Lebesgue measure and the following representation formula holds for the density p_F:

$$p_F(x) = \mathbb{E}\Big(\sum_{i=1}^d \partial_i Q_d(F-x) H_i(F;1)\Big). \tag{5}$$

ii) Suppose in addition that (2) holds for G and $H_i(F,G) \in \cap_{p\in\mathbb{N}} L^p$ for every $i = 1,\dots,d$. Then

$$\begin{array}{rcl} \phi_G(x) & := & \mathbb{E}(G \mid F = x) = 1_{\{p_F>0\}} \dfrac{p_{F,G}(x)}{p_F(x)} \quad \text{with} \\ p_{F,G}(x) & = & \mathbb{E}\Big(\sum_{i=1}^d \partial_i Q_d(F-x) H_i(F;G)\Big). \end{array} \tag{6}$$

Details and proofs can be found in [3] (see Theorem 18 and 19 and their applications to the Wiener space, as in Proposition 22).
We give now a localized version of the above integration by parts formula, for which we need to assume a localized version of the non degeneracy condition (see next condition (7)).

Lemma 3. *Let $F = (F^1,\dots,F^d), G, \Theta$ with $F^i \in \mathbb{D}^{2,\infty}$ and $G, \Theta \in \mathbb{D}^{1,\infty}$ and let $\psi : \mathbb{R} \to \mathbb{R}_+$ be a C_b^1 function such that $\psi(x)\,|(\ln\psi)'(x)|^p$ is bounded for some $p \geq 1$. Assume that*

$$\Gamma_q(F,\Theta) := 1 + (\mathbb{E}((\det\sigma_F)^{-q} 1_{\{\psi(\Theta)\neq 0\}}))^{1/q} < \infty, \quad \text{for every } q. \tag{7}$$

Then, for every test function $f \in C_c^\infty(\mathbb{R}^d)$ one has

$$\mathbb{E}(\nabla f(F) G\psi(\Theta)) = \mathbb{E}(f(F) H(F, G\psi(\Theta))) \tag{8}$$

with $H(F,G\psi(\Theta)) = \delta(G\psi(\Theta)\widehat{\sigma}_F DF)$, i.e. as $i = 1,\dots,d$,

$$\begin{aligned} H_i(F, G\psi(\Theta)) = \psi(\Theta) \sum_{j=1}^d \Big(\; & G\widehat{\sigma}_F^{ji} LF^j - G\langle D\widehat{\sigma}_F^{ji}, DF^j\rangle - \widehat{\sigma}_F^{ji}\langle DG, DF^j\rangle + \\ & -(\ln\psi)'(\Theta) G\widehat{\sigma}_F^{ji}\langle D\Theta, DF^j\rangle\Big). \end{aligned} \tag{9}$$

Moreover, for every $p \geq 1$ there exists universal non negative constants $C_{p,d}, q_{p,d}, \ell_{p,d}$ such that

$$\|H(F, G\psi(\Theta))\|_p \leq C_{p,d} \times \Gamma^2_{q_{p,d}}(F,\Theta) \times \|G\psi(\Theta)\|_{1,q_{p,d}} \times (1 + \|F\|_{2,q_{p,d}})^{\ell_{p,d}}. \tag{10}$$

The proof of Lemma 3 follows by applying the integration by parts formula as in Proposition 1 with G replaced by $G\psi(\Theta)$, so it is immediate and we skip it. Notice that estimate (10) straightforwardly follows from (9) using Meyer's inequality. Finally, in the following we will take a special function ψ, see (15).

Setting

$$d\overline{\mathbb{P}} = \psi(\Theta)d\mathbb{P},$$

we can give an interpretation of the integration by parts formula (8) in terms of $\overline{\mathbb{P}}$. We denote

$$C_p(\psi) = 1 + \sup_x \psi(x) \left|(\ln\psi)'(x)\right|^p \tag{11}$$

and by $\overline{\mathbb{E}}$ the expectation (integral) with respect to $\overline{\mathbb{P}}$.

Lemma 4. *i) If $C_p(\psi) < \infty$ for some $p \geq 1$, then under the hypothesis of Lemma 3 one has*

$$\overline{\mathbb{E}}(\nabla f(F)G) = \overline{\mathbb{E}}(f(F)\overline{H}(F,G)) \tag{12}$$

with

$$\begin{aligned}\overline{H}(F,G) = 1_{\{\psi(\Theta)\neq 0\}} \Big(& G\widehat{\sigma}_F LF - G\left\langle D\widehat{\sigma}_F, DF\right\rangle - \widehat{\sigma}_F\left\langle DG, DF\right\rangle \\ & +(\ln\psi)'(\Theta)G\widehat{\sigma}_F\left\langle D\Theta, DF\right\rangle \Big).\end{aligned}$$

Moreover,

$$\overline{\mathbb{E}}\left|\overline{H}(F,G)\right|^p \leq C_{p,d}\Gamma^2_{q_{p,d}}(F,\Theta)\left\|G\right\|_{1,q_{p,d}}(1+\left\|F\right\|_{2,q_{p,d}})^{\ell_{p,d}}(1+C_p(\psi))(1+\left\|\Theta\right\|_{1,q_{p,d}}). \tag{13}$$

ii) Suppose that (11) holds for some $p > d$. Then the law of F under $\overline{\mathbb{P}}$ is absolutely continuous with respect to the Lebesgue measure, with density $\bar{p}_F$ given by

$$\overline{p}_F(x) = \overline{\mathbb{E}}\Big(\sum_{i=1}^d \partial_i Q_d(F-x)\overline{H}_i(F;1)\Big),$$

and moreover

$$\overline{\mathbb{E}}(|\partial_i Q_d(F-x)|^{\frac{p}{p-1}}) \leq C_{p,d}C_p(\psi)\Big(\Gamma_{q_{d,q}}(F,\Theta)(1+\left\|F\right\|_{2,q_{p,q}})(1+\left\|\Theta\right\|_{1,q_{p,d}})\Big)^{\ell_{p,d}} \tag{14}$$

for every $x \in \mathbb{R}^d$ and $i = 1, \ldots, d$.

Proof. The point *i)* is an immediate consequence of Lemma 3. As for *ii)*, we take $G = 1$ and we notice that (12) means that (2) holds with $G = 1$ under $\overline{\mathbb{P}}$. So (14) follows from Proposition 2. $\square$

3 Small perturbations of a Gaussian random variable

3.1 Preliminary estimates

We consider here a r.v. of the type $F = x + G + R \in \mathbb{R}^d$ where $R \in \mathbb{D}^{2,\infty}$ and

$$G = \sum_{j=1}^\infty \int_0^\infty h_j(s)dW_s^j$$

with $h_j : [0,+\infty) \to \mathbb{R}^d$ deterministic and square integrable. Then G is a centered Gaussian random variable of covariance matrix

$$M_G^{k,p} = \int_0^\infty \langle h^k(s), h^p(s)\rangle ds = \sum_{j=1}^\infty \int_0^\infty h_j^k(s) h_j^p(s) ds.$$

We assume that M_G is invertible and we denote by g_{M_G} the density of G that is

$$g_{M_G}(y) = \frac{1}{(2\pi)^{d/2}\sqrt{\det M_G}} \exp(-\left\langle M_G^{-1}y, y\right\rangle).$$

Our aim is to give estimates of the density of F in terms of g_{M_G}. We will use the following localization function: for $a > 0$, we define $\psi_a : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\psi_a(x) = 1_{[0,a)}(x) + 1_{[a,2a)}(x) \exp\Big(1 - \frac{a}{2a-x}\Big). \tag{15}$$

Then ψ_a is differentiable (except for $x = a$) and one has

$$\psi_a'(x) = -1_{[a,2a)}(x) \frac{a}{(2a-x)^2} \psi_a(x).$$

In particular, for every $p \geq 1$

$$\sup_x |\partial_x \ln \psi_a(x)|^p \, \psi_a(x) \leq C_p(\psi_a) := e \times a^{-p} \sup_{y\geq 0}(y^p e^{-y}). \tag{16}$$

In the following we consider

$$\psi = \psi_{1/8}, \quad \Theta = |DR|^2 = \sum_{j=1}^d \sum_{k=1}^\infty \int_0^{+\infty} |D_s^k R^j|^2 ds \quad \text{and} \quad d\overline{\mathbb{P}} = \psi(|DR|^2) d\mathbb{P}. \tag{17}$$

Lemma 5. *i) Under $\overline{\mathbb{P}}$, the law of F is absolutely continuous with respect to the Lebesgue measure and the density $\overline{p}_F$ satisfies*

$$\sup_{y\in R^d} |\overline{p}_F(y) - g_{M_G}(y-x)| \leq \varepsilon(M_G, R)$$

where

$$\varepsilon(M_G, R) := \frac{C_d}{\sqrt{\det M_G}} (1 + \left\|\overline{R}\right\|_{2,q_d})^{\ell_d} \left\|\overline{R}\right\|_{2,q_d} \quad \textit{with } \overline{R} := M_G^{-1/2} R.$$

Here C_d, q_d, ℓ_d are some universal positive constants depending on d only.
ii) If the law of F under $\mathbb{P}$ has a continuous density p_F, then one has

$$p_F(y) \geq g_{M_G}(y-x) - \varepsilon(M_G, R) \quad \textit{for every } y \in \mathbb{R}^d.$$

iii) Suppose now that $(\det \sigma_F)^{-1} \in \cap_{p\geq 1} L^p$. Then the law of F under $\mathbb{P}$ has a continuous density p_F and

$$\sup_{y\in R^d} |p_F(y) - g_{M_G}(y-x)| \leq \varepsilon(M_G, R) \times \Gamma_{q_d}^2(F, 0),$$

the quantity $\Gamma_{q_d}(F, 0)$ being defined through (7).

Proof. $i)$ We assume first that $x=0$ and G is standard normal, that is $M_G=I_d$ the identity matrix. We also denote $A_\psi=\{\psi(|DR|^2)\neq 0\}$.

Step 1. [Taylor expansion] On the set A_ψ one has $|DR|^2\leq \frac{1}{4}$ so that, if $|\xi|=1$ then

$$
\begin{aligned}
\langle\sigma_F\xi,\xi\rangle &= \int_0^\infty \langle D_sF,\xi\rangle^2\,ds \geq \frac{1}{2}\int_0^\infty \langle D_sG,\xi\rangle^2\,ds-\int_0^\infty \langle D_sR,\xi\rangle^2\,ds\\
&\geq \frac{1}{2}-\int_0^\infty |D_sR|^2\,ds=\frac{1}{2}-|DR|^2\geq\frac{1}{4}.
\end{aligned}
$$

As a consequence of $ii)$ in Lemma 4, the law of F under $\overline{\mathbb{P}}$ is absolutely continuos with respect to the Lebesgue measure and the density $\overline{p}_F$ is given by

$$
\overline{p}_F(y)=\sum_{j=1}^d \overline{\mathbb{E}}(\partial_j Q_d(F-y)\overline{H}_j).
$$

with

$$
\overline{H}_j=1_{A_\psi}\sum_{\ell=1}^d\Big(\widehat{\sigma}_F^{\ell j}LF^\ell-\langle D\widehat{\sigma}_F^{\ell j},DF^\ell\rangle+(\ln\psi)'(|DR|^2)\widehat{\sigma}_F^{\ell j}\langle D|DR|^2,DF^\ell\rangle\Big).
$$

We denote $F_\eta=G+\eta R$, $\eta\in[0,1]$, and we write

$$
\begin{aligned}
\overline{p}_F(y) &=\sum_{j=1}^d \overline{\mathbb{E}}(\partial_j Q_d(G-y)\overline{H}_j)+S(y)\quad\text{with}\\
S(y) &:=\sum_{k,j=1}^d\int_0^1 \overline{\mathbb{E}}(\partial_k\partial_j Q_d(F_\eta-y)R_k\overline{H}_j)d\eta.
\end{aligned}
$$

Remark 6. *Notice that $\partial_k\partial_jQ_d(F_\eta-y)$ is well defined except for $F_\eta=y$. And since the law of F_η is absolutely continuous this singularity is not a problem. But we also need some integrability properties and this is less easy to check. In fact, for a completely rigorous computation we have to replace the kernel Q_d by a truncated kernel Q_d^ε, to achieve all the computations for Q_d^ε and then to pass to the limit with* $\varepsilon\to 0$. *Since the upper bounds that we obtain in the sequel do not depend on ε the result will follow. So we will skip this technical difficulty.*

Step 2. [Estimate of the remainder] By using arguments similar to the ones developed above, on the set A_ψ one has $\inf_{|\xi|=1}\langle\sigma_{F_\eta}\xi,\xi\rangle\geq\frac{1}{4}$, so that $\Gamma_p(F_\eta,|DR|^2)\leq 1+4^d$. Using the integration by parts formula (12) with $G_{j,k}=R_k\overline{H}_j$ and $\Theta=|DR|^2$ we obtain

$$
\overline{\mathbb{E}}(\partial_k\partial_jQ_d(F_\eta-y)R_k\overline{H}_j)=\overline{\mathbb{E}}(\partial_jQ_d(F_\eta-y)\overline{H}_{j,k})
$$

with

$$
\begin{aligned}
\overline{H}_{j,k} &= 1_{A_\psi}\Big(G_{j,k}\widehat{\sigma}_{F_\eta}LF_\eta-G_{j,k}\langle D\widehat{\sigma}_{F_\eta},DF_\eta\rangle-\widehat{\sigma}_{F_\eta}\langle DG_{j,k},DF_\eta\rangle\\
&\qquad+(\ln\psi)'(|DR|^2)G_{j,k}\widehat{\sigma}_{F_\eta}\langle D|DR|^2,DF_\eta\rangle\Big).
\end{aligned}
$$

We take $p = d + 1$ and we use Hölder's inequality in order to get

$$\left|\overline{\mathbb{E}}(\partial_j Q_d(F_\eta - y)\overline{H}_{j,k})\right| \le \overline{\mathbb{E}}\Big(|\partial_j Q_d(F_\eta - y)|^{p/(p-1)}\Big)^{(p-1)/p} \|\overline{H}_{j,k}\|_p.$$

Moreover, by (13)

$$\begin{aligned}\|\overline{H}_{j,k}\|_p &\le C_d(1 + \|F_\eta\|_{2,q_d})^{\ell_d}(1 + \|\,|DR|^2\,\|_{1,q_d})^{\ell_d}\|R_k\overline{H}_j\|_{1,q_d} \\ &\le C'_d(1 + \|R\|_{2,q_d})^{\ell'_d}\|R\|_{1,2q_d}.\end{aligned}$$

The second inequality is true because $\|\overline{H}_j\|_{1,2q_d}$ may be estimated using (13) again. Moreover, (14) and the same estimates as above give $\overline{\mathbb{E}}(|\partial_j Q_d(F_\eta - y)|^{p/(p-1)}) \le C_d(1 + \|R\|_{2,q_d})^{\ell_d}$ and finally we have proved that

$$|S(y)| \le C_d(1 + \|R\|_{2,q_d})^{\ell_d}\,\|R\|_{1,q_d}\,. \tag{18}$$

Step 3. [The main term and its rewriting in terms of G only] We write now

$$\overline{\mathbb{E}}(\partial_j Q_d(G - y)\overline{H}_j) = \mathbb{E}(\partial_j Q_d(G - y)LG) + U_j(y)$$

with

$$U_j(y) := \overline{\mathbb{E}}\big(\partial_j Q_d(G - y)(J_1 - J_2 + J_3)\big) - \mathbb{E}(\partial_j Q_d(G - y)J_4)$$

and (recall that I_d is the identity d dimensional matrix):

$$\begin{aligned} J_1 &= 1_{A_\psi}(\widehat{\sigma}_F LF - I_d LG), & J_2 &= 1_{A_\psi}\langle D\widehat{\sigma}_F, DF\rangle \\ J_3 &= 1_{A_\psi}(\ln\psi)'(|DR|^2)\widehat{\sigma}_F\langle D\,|DR|^2\,, DF\rangle, & J_4 &= (1 - \psi(|DR|^2))LG \end{aligned}$$

We will prove that for every positive integer q there exist some universal constants $C_{q,d}, \ell_{q,d}$ such that

$$\begin{aligned}&(\overline{\mathbb{E}}(|J_i|^q))^{1/q} \le C_{q,d}(1 + \|R\|_{2,q})^{\ell_{q,d}}\|R\|_{2,q}, \quad i = 1, 2, 3 \text{ and} \\ &(\mathbb{E}(|J_4|^q))^{1/q} \le C_{q,d}(1 + \|R\|_{2,q})^{\ell_{q,d}}\|R\|_{2,q}.\end{aligned} \tag{19}$$

In the following, the notation C_d will stand for a suitable positive constant depending on d only.
Since $\det\sigma_F \ge \frac{1}{4}$ on the set A_ψ a straightforward computation gives

$$\begin{aligned}|(\widehat{\sigma}_F - I_d)^{ij}| = |(\widehat{\sigma}_{G+R} - \widehat{\sigma}_G)^{ij}| &\le C_d \max_{k,p}|\sigma_F^{k,p}|^{d-1}\big(1 + |DR|\big)|DR| \\ &\le C_d\big(1 + |DR|\big)^{2d-1}|DR|.\end{aligned}$$

One also has (Meyer's inequality)

$$\begin{aligned}&\overline{\mathbb{E}}(|LF|^q) \le \mathbb{E}(|LF|^q) \le C\,\|F\|_{2,q}^q \le C(1 + \|R\|_{2,q}^q), \\ &\overline{\mathbb{E}}(|LR|^q) \le \mathbb{E}(|LR|^q) \le C\,\|R\|_{2,q}^q\end{aligned}$$

so using Hölder inequality we obtain (19), for $i=1$.
We estimate now J_2. Notice first that $|D\sigma_F^{ij}| \leq C(|DR| + \left|D^{(2)}R\right|)^2$. We denote by $\widetilde{\sigma}_F^{ij}$ the matrix of cofactors for σ_F, that is $\widehat{\sigma}_F = (\det\sigma_F)^{-1}\widetilde{\sigma}_F$, with $\widetilde{\sigma}_F^{ij} = \det(\mathrm{minor}(\sigma_F, j, i))$, $\mathrm{minor}(\sigma_F, j, i)$ being the $(d-1)\times(d-1)$ minor of σ_F obtained by deleting the jth row and the ith column of σ_F. Since on the set A_ψ one has $\det\sigma_F \geq 1/4$ we obtain

$$\begin{aligned}
|D\widehat{\sigma}_F^{i,j}| &\leq C_d\left(|D\det\sigma_F| + |D\widetilde{\sigma}_F^{ij}|\right)\\
&\leq C_d\Big(1+|DF|+|D^{(2)}F|\Big)^{2(d-1)}\sum_{i,j=1}^d |D\sigma_F^{ij}|\\
&\leq C_d'\Big(1+|DR|+|D^{(2)}R|\Big)^{2(d-1)}\Big(|DR|+|D^{(2)}R|\Big)^2
\end{aligned}$$

from which (19) for $i=2$ follows. And (19) is straightforward for $i=3$.
Let us now estimate J_4. Since $LG = G$ we may use Chebycev's inequality and we obtain

$$\mathbb{E}(|J_4|^q) \leq C_q\mathbb{P}\Big(|DR|^2 \geq \frac{1}{4}\Big) \leq C_q\|R\|_{1,q}.$$

For $p = d+1 > d$ we have $\overline{\mathbb{E}}(|\partial_j Q_d(G-y)|^{p/(p-1)}) \leq C_d$ and $\mathbb{E}(|\partial_j Q_d(G-y)|^{p/(p-1)}) \leq C_d$, so using Hölder's inequality

$$|U_j(y)| \leq C_d(1+\|R\|_{2,q_d})^{\ell_d}\,\|R\|_{2,q_d}\,.$$

Finally

$$\sum_{j=1}^d \mathbb{E}(\partial_j Q_d(G-y)LG) = g_{I_d}(y)$$

where g_{I_d} is the density of the standard normal variable. Therefore, we get

$$\overline{p}_F(y) = g_{I_d}(y) + S(y) + U(y) \quad \text{and} \quad |S(y)|+|U(y)| \leq C_d(1+\|R\|_{2,q_d})^{\ell_d}\,\|R\|_{1,q_d}\,.$$

Step 4. [The general case] Consider now a general random variable of the form $F = x+G+R$ and let M_G denote the covariance matrix of G. Set $\overline{G} = M^{-1/2}G, \overline{R} = M^{-1/2}R$ and $\overline{F} = \overline{G}+\overline{R}$, so that $F = x+M^{1/2}\overline{F}$, and set $\overline{p}_{\overline{F}}$ the density of $\overline{F}$ under $\overline{\mathbb{P}}$. By taking $f_\delta(z) = (2\delta)^{-d}1_{B_\delta(y)}(z)$, we can write

$$\begin{aligned}
\overline{\mathbb{E}}(f_\delta(F)) &= \overline{\mathbb{E}}(f_\delta(x+M_G^{1/2}\overline{F})) = \int f_\delta(x+M_G^{1/2}z)\overline{p}_{\overline{F}}(z)dz\\
&= \int f_\delta(x+M_G^{1/2}z)g_{I_d}(z)dz + \int f_\delta(x+M_G^{1/2}z)(S+U)(z)dz\\
&= \int f_\delta(z)g_{M_G}(z-x)dz + \frac{1}{\det M_G^{1/2}}\int f_\delta(z)(S+U)(M_G^{1/2}(z-x))dz.
\end{aligned}$$

This yields

$$\left|\overline{\mathbb{E}}(f_\delta(F)) - \int f_\delta(z)g_{M_G}(z-x)dz\right| \leq \frac{C_d}{\sqrt{\det M_G}}(1+\left\|\overline{R}\right\|_{2,q_d})^{\ell_d}\left\|\overline{R}\right\|_{1,q_d} = \varepsilon(M_G,R).$$

As $\delta \to 0$, we obtain the same inequality for $|\overline{p}_F(y) - g_{M_G}(y-x)|$. So the point $i)$ is proved.
Step 5. [**Proof of** $ii)$] We write

$$\mathbb{E}(f_\delta(F)) \geq \overline{\mathbb{E}}(f_\delta(F)) \geq \int f_\delta(y) g_{M_G}(y-x)dy - \varepsilon(M_G, R).$$

and passing to the limit with $\delta \to 0$ we obtain $p_F(y) \geq g_{M_G}(y-x) - \varepsilon(M_G, R)$, and this gives $ii)$.
Step 6. [**Proof of** $iii)$] We write

$$\begin{aligned}
\Big|\mathbb{E}(f_\delta(F)) - \int f_\delta(z) g_{M_G}(z)dz\Big| &\leq \Big|\overline{\mathbb{E}}(f_\delta(F)) - \int f_\delta(z) g_{M_G}(z)dz\Big| + \\
&\quad + \big|\mathbb{E}(f_\delta(F)(1 - \psi(\big|D\overline{R}\big|^2)))\big| \\
&\leq \varepsilon(M_G, \overline{R}) + \Big|\mathbb{E}\Big(f_\delta(F)\mathbb{E}\big(1 - \psi(|D\overline{R}|^2)\big|F\big)\Big)\Big|
\end{aligned}$$

and using f_δ as above this gives

$$|p_F(y) - g_{M_G}(y)| \leq \varepsilon_p(M_G, \overline{R}) + p_F(y)\big|\mathbb{E}(1 - \psi(|D\overline{R}|^2) \mid F = y)\big|.$$

Let $\Lambda := 1 - \psi(|D\overline{R}|^2)$. Using Proposition 2 we have

$$\begin{aligned}
p_F(y)\left|\mathbb{E}\big(1 - \psi(|D\overline{R}|^2) \mid F = y\big)\right| &= \Big|\mathbb{E}\Big(\sum_{i=1}^d \partial_i Q_d(F-y) H_i(F, \Lambda)\Big)\Big| \\
&\leq C_d \max_{i=1,\ldots,d} \Big(\mathbb{E}|H_i(F,\Lambda)|^p\Big)^{1/p} \\
&\leq C_{p,d}\Gamma^2_{q_d}(F,1) \left\|\Lambda\right\|_{1,q_d} (1 + \left\|F\right\|_{2,q_q})^{\ell_d}
\end{aligned}$$

the last inequality being a consequence of (10). Since $\left\|\Lambda\right\|_{1,q_d} \leq \sqrt{\mathbb{P}(\overline{R} \geq \frac{1}{4})}\,(1 + \left\|\overline{R}\right\|_{1,2q_d})$ the proof is now completed. $\square$

3.2 Quantitative estimates

In this section, we consider a time interval of the type $[T-\delta, T]$, where $T > 0$ is a fixed horizon and $0 < \delta \leq T$, and we use the Malliavin calculus with respect to $W_s, s \in [T-\delta, T]$. In particular, we take conditional expectations with respect to $\mathcal{F}_{T-\delta}$. Therefore, for $U = (U^1, ..., U^d)$, $U_i \in \mathbb{D}^{L,p}$, we define the following conditional Malliavin Sobolev norms:

$$\left\|U\right\|^p_{\delta,L,p} = \mathbb{E}(|U|^p \mid \mathcal{F}_{T-\delta}) + \sum_{l=1}^L \mathbb{E}\Big(\Big(\int_{[T-\delta,T]^l} \left|D^{(l)}_{s_1\ldots s_l} U\right|^2 ds_1 \ldots ds_l\Big)^{p/2} \Big| \mathcal{F}_{T-\delta}\Big). \quad (20)$$

Let F denote a d dimensional functional on the Wiener space which is $\mathcal{F}_T$ measurable and assume that for $\delta \in (0, T]$ the following decomposition holds:

$$F = F_{T-\delta} + G_\delta + R_\delta \quad (21)$$

where $F_{T-\delta}$ is an $\mathcal{F}_{T-\delta}$ measurable random variable, $R_\delta \in (\mathbb{D}^{2,\infty})^d$ and

$$G_\delta = \sum_{k=1}^{\infty} \int_{T-\delta}^{T} h_\delta^k(s) dW_s^k.$$

Here $h_\delta^k(s), s \in [T-\delta, T]$ are progressively measurable processes such that $h_\delta^k(s)$ is $\mathcal{F}_{T-\delta}$ measurable and $\sum_{k=1}^{\infty} \int_{T-\delta}^{T} |h_\delta^k(s)|^2 ds < \infty$ a.s. In particular, conditionally on $\mathcal{F}_{T-\delta}$, the random variable G_δ is centered and Gaussian with covariance matrix

$$C_\delta^{ij} = \sum_{k=1}^{\infty} \int_{T-\delta}^{T} h_\delta^{k,i}(s) h_\delta^{k,j}(s) ds \quad 1 \leq i, j \leq d.$$

On the set $\{\det C_\delta \neq 0\} \in \mathcal{F}_{T-\delta}$, we define the (random) norm

$$|x|_\delta := |C_\delta^{-1/2} x|, \quad x \in \mathbb{R}^d$$

and for $q \in \mathbb{N}$, we consider the following quantity:

$$\theta_{\delta,q}^q = \mathbb{E}(|R_\delta|_\delta^q \mid \mathcal{F}_{T-\delta}) + \mathbb{E}\Big(\sum_{l=1}^{2} \Big(\int_{[T-\delta,T]^l} \big|D_{s_1 \ldots s_l}^{(l)} R_\delta\big|_\delta^2 \, ds_1 \ldots ds_l \Big)^{q/2} \Big| \mathcal{F}_{T-\delta} \Big). \tag{22}$$

Notice that by (20), one has

$$\theta_{\delta,q} = \|C_\delta^{-1/2} R_\delta\|_{\delta,2,q} \quad \text{on the set } \{\det C_\delta \neq 0\}. \tag{23}$$

Set now $\overline{\mathbb{P}}_\delta(\omega, \cdot)$ the measure induced by $\overline{\mathbb{E}}_\delta(\omega, X) = \mathbb{E}(X \psi(|DR_\delta|^2) \mid \mathcal{F}_{T-\delta})(\omega)$, where ψ is defined in (17). By developing in a conditional form the arguments as in the proof of Lemma 5, on the set $\{\det C_\delta \neq 0\}$ one gets that under $\overline{\mathbb{P}}_\delta(\omega, \cdot)$ the law of F has a regular density w.r.t. the Lebesgue measure. Therefore, there exists a function $\bar{p}_{F,\delta}(\omega, z)$ which is regular as a function of z and such that

$$\mathbb{E}\big(f(F)\psi(|DR_\delta|^2) \mid \mathcal{F}_{T-\delta}\big)(\omega) = \int f(z) \bar{p}_{F,\delta}(\omega, z) dz, \quad \omega \in \{\det C_\delta \neq 0\} \tag{24}$$

for any measurable and bounded function f.

Now, let us introduce the following set: for $y \in \mathbb{R}^d$ and $r > 0$, we set

$$\begin{aligned} &\Gamma_{\delta,r}(y) = \big\{|F_{T-\delta} - y|_\delta \leq r\big\} \cap \big\{ \det C_\delta \neq 0\big\} \cap \big\{\theta_{\delta,q_d} \leq a(r)\big\}, \quad \text{where} \\ &a(r) = 1 \wedge \frac{1}{C_d 2^{\ell_d+1} (2\pi)^{d/2} e^{r^2}} \end{aligned} \tag{25}$$

and q_d, ℓ_d and C_d are the universal constants defined in (i) of Lemma 5. Then we have

Theorem 7. *For $\delta \in (0, T]$, let decomposition (21) hold and for $y \in \mathbb{R}^d$, $r > 0$, let $\Gamma_{\delta,r}(y)$ be the set in (25).*
i) For every non negative and measurable function $f : \mathbb{R}^d \to \mathbb{R}$ one has

$$\mathbb{E}(f(F)\,|\,\mathcal{F}_{T-\delta})(\omega) \geq \frac{1}{2e^{r^2}(2\pi)^{d/2}}\,(\det C_\delta)^{-1/2}\int f(z)\mathbf{1}_{\Gamma_{\delta,r}(z)}dz.$$

ii) Suppose F has a density p_F. Then for every $y \in \mathbb{R}^d$ such that p_F is continuous in a neighborhood of y and for every $r > 0$ one has

$$p_F(y) \geq \frac{1}{2e^{r^2}(2\pi)^{d/2}}\,\mathbb{E}\big((\det C_\delta)^{-1/2}\mathbf{1}_{\Gamma_{\delta,r}(y)}\big).$$

Proof. Let $\omega \in \{\det C_\delta \neq 0\}$. By using (24), for any measurable and non negative function f we have

$$\begin{aligned}\mathbb{E}(f(F)\mathcal{F}_{T-\delta})(\omega) &\geq \mathbb{E}(f(F)\psi(|DR_\delta|^2)\mid\mathcal{F}_{T-\delta})(\omega) = \int f(z)\overline{p}_{F,\delta}(\omega,z)dz\\ &\geq \int f(z)\overline{p}_{F,\delta}(\omega,z)\mathbf{1}_{\Gamma_{\delta,r}(z)}dz\end{aligned}$$

Using Lemma 5 in a conditional form (with respect to $\mathcal{F}_{T-\delta}$) we obtain

$$\overline{p}_{F,\delta}(\omega,z) \geq g_{C_\delta(\omega)}(z - F_{T-\delta}(\omega)) - \varepsilon(C_\delta(\omega), R_\delta)(\omega)$$

where, by using (22),

$$\begin{aligned}\varepsilon(C_\delta, R_\delta)(\omega) &\leq \frac{C_d}{\sqrt{\det C_\delta}}(1 + \|C_\delta^{-1/2}R_\delta\|_{(\delta),2,q_d})^{\ell_q}\|C_\delta^{-1/2}R_\delta\|_{(\delta),2,q_d}\\ &= \frac{C_d}{\sqrt{\det C_\delta}}(1+\theta_{\delta,q_d})^{\ell_q}\theta_{\delta,q_d}.\end{aligned}$$

If $\omega \in \Gamma_{\delta,r}(z)$ then $\theta_{\delta,q_d} \leq a(r) \leq 1$ so that

$$\varepsilon(C_\delta, R_\delta)(\omega) \leq \frac{C_d}{\sqrt{\det C_\delta}}2^{\ell_q}a(r).$$

For $\omega \in \Gamma_{\delta,r}(z)$ we also have

$$\big\langle C_\delta^{-1}(F_{T-\delta} - z), F_{T-\delta} - z\big\rangle = |F_{T-\delta} - z|_\delta^2 \leq r^2$$

so that

$$g_{C_\delta}(z - F_{T-\delta}) \geq \frac{1}{(2\pi)^{d/2}\sqrt{\det C_\delta}}e^{-r^2}.$$

Then, by the choice of $a(r)$ we obtain

$$\overline{p}_{F,\delta}(\omega,z) \geq \frac{1}{(2\pi)^{d/2}\sqrt{\det C_\delta}}e^{-r^2} - \varepsilon(C_\delta, R_\delta)(\omega) \geq \frac{1}{2(2\pi)^{d/2}\sqrt{\det C_\delta}}e^{-r^2}.$$

We conclude that

$$\mathbb{E}(f(F)\mid \mathcal{F}_{T-\delta})(\omega)\geq \frac{1}{2e^{r^2}(2\pi)^{d/2}}\int f(z)(\det C_\delta)^{-1/2}1_{\Gamma_{\delta,r}(z)}dz$$

and i) is proved. As for ii), we write

$$\mathbb{E}(f(F))=\mathbb{E}(\mathbb{E}(f(F)\mid \mathcal{F}_{T-\delta}))\geq \frac{1}{2(2\pi)^{d/2}}e^{-r^2}\times\int f(z)\mathbb{E}(\frac{1}{\sqrt{\det C_\delta}}1_{\Gamma_{\delta,r}(z)})dz.$$

Since the inequality holds for every non negative function f, the statement holds. $\square$

4 Examples

We apply now Theorem 7 to two cases in which a support theorem is available and we give results for the strict positivity and lower bounds for the density which involve suitable local or global non degeneracy conditions on the skeleton.

4.1 Ito processes

We consider here a process $Z_t=(X_t,Y_t)^*$, taking values on $\mathbb{R}^d\times\mathbb{R}^n$, which solves the following stochastic differential equation: as $t\leq T$,

$$\begin{aligned} X_t=&\quad x_0+\sum_{j=1}^m\int_0^t\sigma_j(X_t,Y_t)dW_t^j+\int_0^t b(X_t,Y_t)dt\\ Y_t=&\quad y_0+\sum_{j=1}^m\int_0^t\alpha_j(X_t,Y_t)dW_t^j+\int_0^t \beta(X_t,Y_t)dt. \end{aligned}\tag{26}$$

We are interested in dealing with strict positivity and/or lower bounds for the probability density function of one component at a fixed time, say X_T, as a consequence of Theorem 7. As for the positivity property, this is a case in which a support theorem is available, and we are going to use it strongly. For diffusion processes, we get an example which is essentially the same as in the paper of Ben Arous and Leandre [7] and in the paper of Aida, Kusouka and Stroock [1]. Concerning the lower bounds, we will use lower estimates for the probability that Ito processes stays in a tube around a path proved by Bally, Fernández and Meda in [5].

So, in (26) we assume that $\sigma_j,b\in C_b^4(\mathbb{R}^{d+n};\mathbb{R}^d)$ and $\alpha_j,\beta\in C_b^4(\mathbb{R}^{d+n};\mathbb{R}^n)$, $j=1,\dots,m$, which amounts to say that $X_t^\ell,Y_t^i\in\mathbb{D}^{2,\infty}$ for all ℓ and i.

For $\phi\in L^2([0,T];\mathbb{R}^m)$, let $z_t(\phi)=(x_t(\phi),y_t(\phi)^*$ denote the skeleton associated to (26), i.e.

$$\begin{aligned} x_t(\phi)=&\quad x_0+\sum_{j=1}^m\int_0^t\sigma_j\big(x_t(\phi),y_t(\phi)\big)dW_t^j+\int_0^t \overline{b}\big(x_t(\phi),y_t(\phi)\big)dt\\ y_t(\phi)=&\quad y_0+\sum_{j=1}^m\int_0^t\alpha_j\big(x_t(\phi),y_t(\phi)\big)dW_t^j+\int_0^t \overline{\beta}\big(x_t(\phi),y_t(\phi)\big)dt, \end{aligned}\tag{27}$$

in which $\overline{b} = b - \frac{1}{2}\sum_{j=1}^m \partial_{\sigma_j}\sigma_j$ and $\overline{\beta} = \beta - \frac{1}{2}\sum_{j=1}^m \partial_{\alpha_j}\alpha_j$, where we have used the notation $(\partial_g f)^i = g\cdot\nabla f^i$.

For a fixed $x\in\mathbb{R}^d$, we also set

$$\mathcal{C}(x) = \{\phi\in L^2([0,T];\mathbb{R}^m)\,:\, x_T(\phi) = x\}. \tag{28}$$

We finally consider the following set: for fixed $\mu\geq 1$ and $h>0$,

$$L(\mu,h) = \{f\,:\,[0,T]\to\mathbb{R}_+\,;\, f_t\leq \mu f_s \text{ for all } t,s \text{ such that } |t-s|\leq h\}. \tag{29}$$

We have

Theorem 8. *Let $Z=(X,Y)^*$ denote the solution of (26), with $\sigma_j, b\in C_b^4(\mathbb{R}^{d+n};\mathbb{R}^d)$ and $\alpha_j,\beta\in C_b^4(\mathbb{R}^{d+n};\mathbb{R}^n)$, $j=1,\ldots,m$. Let $x\in\mathbb{R}^d$ be fixed and suppose that $\mathcal{C}(x)\neq\emptyset$. For $\phi\in\mathcal{C}(x)$, let $z_t(\phi) = (x_t(\phi), y_t(\phi))^*$ be as in (27).*

i) Suppose there exists $\phi\in\mathcal{C}(x)$ such that $\sigma\sigma^*(x, y_T(\phi))>0$. If the law of X_T admits a local density p_{X_T} in x one has

$$p_{X_T}(x) > 0.$$

ii) Suppose there exists $\phi\in\mathcal{C}(x)$ such that $|\partial x_t(\phi)|\in L(\mu,h)$, for some $\mu\geq 1$ and $h>0$, and

$$\sigma\sigma^*\big(x_t(\phi), y\big)\geq \lambda_* > 0 \quad \textit{for all } t\in[0,T] \textit{ and } y\in\mathbb{R}^n.$$

Then the law of X_T admits a local density p_{X_T} in x and one has

$$p_{X_T}(x)\geq \Upsilon\exp\Big[-Q\Big(\Theta + \frac{1}{\lambda_*}\int_0^T|\partial_t x_t(\phi)|dt\Big)\Big]$$

where Υ, Q, Θ are all positive constants depending on d,T,ϕ,μ,h,λ_* and vector fields σ_j,α_j, $j=1,\ldots,m$, and b,β.

It should be noticed that in point *i)* we have to require the existence of a local density because this does follow from the hypotheses $\sigma\sigma^*(x,y_T(\phi))>0$ for some $\phi\in\mathcal{C}(x)$. On the contrary, the strongest requirement in *ii)* does allow quite immediately to assert that a local density exists. Moreover, such requirement can be reformulated in order to be independent of the skeleton, see next Proposition 9.

Proof of Theorem 8. For $0<\delta\leq T$, we consider the decomposition $X_T = X_{T-\delta} + G_\delta + R_\delta$, where

$$G_\delta = \sum_{j=1}^m\int_{T-\delta}^T \sigma_j(X_{T-\delta}, Y_{T-\delta})dW_t^j$$

$$R_\delta = \sum_{j=1}^m\int_{T-\delta}^T\big(\sigma_j(X_t,Y_t) - \sigma_j(X_{T-\delta},Y_{T-\delta})\big)dW_t^j + \int_{T-\delta}^T b(X_t,Y_t)dt.$$

Conditionally on $\mathcal{F}_{T-\delta}$, the covariance matrix of the Gaussian r.v. G_δ is

$$C_\delta = \sigma\sigma^*(X_{T-\delta}, Y_{T-\delta})\delta.$$

So, we are in the same framework studied in Section 3 and we proceed in order to apply Theorem 7.

i) For $\phi \in \mathcal{C}(x)$, we denote $z^\phi(x) = (x, y_T(\phi))$ and we take ϕ such that $\sigma\sigma^*(z^\phi(x)) > 0$. Then, there exists $\eta > 0$ such that

$$\lambda^* I_d \geq \sigma\sigma^*(z) \geq \lambda_* I_d \quad \text{for every } z \text{ such that } |z - z^\phi(x)| < \eta.$$

For a fixed $\delta \in (0, T]$, we have $|z^\phi(x) - z_{T-\delta}(\phi)| = |z_T(\phi) - z_{T-\delta}(\phi)| \leq C(1 + \|\phi\|_2)\sqrt{\delta} = C_\phi\sqrt{\delta}$, so that if $|Z_{T-\delta} - z_{T-\delta}(\phi)| < C_\phi\sqrt{\delta}$ then $|Z_{T-\delta} - z^\phi(x)| < 2C_\phi\sqrt{\delta}$. We choose δ_0 such that $2C_\phi\sqrt{\delta} < \eta$ for all $\delta < \delta_0$. So, if $|Z_{T-\delta} - z_{T-\delta}(\phi)| < C_\phi\sqrt{\delta}$ we get

$$\lambda_*\delta I_d \leq C_\delta \leq \lambda^*\delta I_d$$

and in particular,

$$\begin{aligned}
|X_{T-\delta} - x|_\delta = |C_\delta^{-1/2}(X_{T-\delta} - x)| &\leq \frac{1}{\sqrt{\lambda_*\delta}}\,|X_{T-\delta} - x| \\
&\leq \frac{1}{\sqrt{\lambda_*\delta}}\,|Z_{T-\delta} - z^\phi(x)| < \frac{2C_\phi}{\sqrt{\lambda_*}}.
\end{aligned}$$

Moreover, for $q \geq 2$, a standard reasoning gives

$$\begin{aligned}
\|R_\delta\|_{\delta,2,q}^q &= \mathbb{E}(|R_\delta|^q \mid \mathcal{F}_{T-\delta}) + \mathbb{E}\Big(\sum_{l=1}^{2}\Big(\int_{[T-\delta,T]^l} \big|D^{(l)}_{s_1\ldots s_l}R_\delta\big|^2\, ds_1\ldots ds_l\Big)^{q/2} \mid \mathcal{F}_{T-\delta}\Big) \\
&\leq (C_{1,q}\delta)^q,
\end{aligned}$$

so that

$$\theta_{\delta,q} = \|C_\delta^{-1/2}R_\delta\|_{\delta,2,q} \leq \frac{1}{\sqrt{\lambda_*\delta}}\,\|R_\delta\|_{\delta,2,q} \leq C_{2,q}\sqrt{\delta}. \tag{30}$$

We set now $r = 2C_\phi/\sqrt{\lambda_*}$ and we take $\delta < \delta_0$ in order that $C_{2,q}\sqrt{\delta} < a(r)$. For such a δ we get that if $\omega \in \{|Z_{T-\delta} - z_{T-\delta}(\phi)| < C_\phi\sqrt{\delta}\}$ then $C_\delta(\omega) \geq \lambda_*\delta I_d$ and $\omega \in \Gamma_{\delta,r}(x)$. Then, by using Theorem 7 we obtain

$$\begin{aligned}
p_{X_T}(x) &\geq \frac{1}{2(2\pi)^{d/2}e^{r^2}}\mathbb{E}\big((\det C_\delta)^{-1/2}\mathbf{1}_{\{|Z_{T-\delta} - z_{T-\delta}(\phi)| < C_\phi\sqrt{\delta}\}}\big) \\
&\geq \frac{1}{2(2\pi\lambda_*\delta)^{d/2}e^{r^2}}\mathbb{P}\big(|Z_{T-\delta} - z_{T-\delta}(\phi)| < C_\phi\sqrt{\delta}\big).
\end{aligned}$$

Now, by the support theorem we have $\mathbb{P}(|Z_{T-\delta} - z_{T-\delta}(\phi)| < \varepsilon) > 0$ for every $\varepsilon > 0$, in particular for $\varepsilon = C_\phi\sqrt{\delta}$, so that $p_{X_T}(x) > 0$.

$ii)$ For $\xi : [0,T] \to \mathbb{R}^d$ and $R > 0$, we set

$$\tau_R^\phi(\xi) = \inf\{t : |\xi_t - x_t(\phi)| \geq R\}.$$

We know that there exists $\phi \in \mathcal{C}(x)$ and $\eta > 0$ such that if $\tau_\eta^\phi(\xi) > T$ then

$$\lambda^* I_d \geq \sigma\sigma^*(\xi_t, y) \geq \lambda_* I_d$$

for any $t \in [0,T]$ and $y \in \mathbb{R}^n$. So, on the set $\{\tau_\eta^\phi(X) > T\}$ one gets $\lambda^*\delta I_d \geq C_\delta \geq \lambda_*\delta I_d$. Moreover, if $\tau_\eta^\phi(X) > T$ then for $0 < \delta < T$

$$\begin{aligned} |X_{T-\delta} - x| = |X_{T-\delta} - x_T(\phi)| &\leq |X_{T-\delta} - x_{T-\delta}(\phi)| + |x_{T-\delta}(\phi) - x_T(\phi)| \\ &< \eta + \int_{T-\delta}^T |\partial_t x_t(\phi)| dt. \end{aligned}$$

Since again (30) holds, we take $\delta < T$ such that $\int_{T-\delta}^T |\partial_t x_t(\phi)| dt < \eta$ and $\theta_{\delta,q_d} \leq a(2\eta)$. Therefore, $\{\tau_\eta^\phi(X) > T\} \subset \Gamma_{\delta,2\eta}(x)$ and by using Theorem 7 we get

$$p_{X_T}(x) \geq \frac{1}{2(2\pi\lambda_*\delta)^{d/2} e^{4\eta^2}} \mathbb{P}\big(\tau_\eta^\phi(X) > T\big) \equiv \Upsilon \times \mathbb{P}\big(\tau_\eta^\phi(X) > T\big).$$

Now, the required hypothesis allow one to use Theorem 1, pg. 14, of Bally, Fernández and Meda [5]: one has

$$\mathbb{P}(\tau_\eta^\phi(X) > T) \geq \exp\Big(- Q\Big(\Theta + \frac{1}{\lambda_*}\int_0^T |\partial_t x_t(\phi)| dt\Big)\Big)$$

and the statement holds. $\square$

Part $ii)$ of Theorem 8 can be reformulated in a more practical way. In fact, one has

Proposition 9. *Let $Z = (X,Y)^*$ denote the solution of (26), with $\sigma_j, b \in C_b^4(\mathbb{R}^{d+n}; \mathbb{R}^d)$ and $\alpha_j, \beta \in C_b^4(\mathbb{R}^{d+n}; \mathbb{R}^m)$, $j = 1, \ldots, m$. Let $x \in \mathbb{R}^d$ be fixed and suppose that there exists a path $x_t \in C^1([0,T], \mathbb{R}^d)$ such that x_0 is as in (26), $x_T = x$, $|\partial x_t| \in L(\mu, h)$, for some $\mu \geq 1$ and $h > 0$, and*

$$\sigma\sigma^*\big(x_t, y\big) \geq \lambda_* > 0 \quad \textit{for all } t \in [0,T] \textit{ and } y \in \mathbb{R}^n.$$

Then the law of X_T admits a local density p_{X_T} in x and one has

$$p_{X_T}(x) \geq \Upsilon \exp\Big[- Q_d\Big(\Theta_T + \frac{1}{\lambda_*}\int_0^T |\partial_t x_t(\phi)| dt\Big)\Big]$$

where Υ, Q_d, Θ_T are all positive constants depending on $d, T, \phi, \mu, h, \lambda_*$ and the diffusion coefficients $\sigma_j, b, \alpha_j, \beta$.

Proof. The statement immediately follows from $ii)$ of Theorem 8 once we prove that there exists $\phi \in \mathcal{C}(x)$ such that $x_t = x_t(\phi)$, $t \in [0, T]$. In fact, set y_t the solution of

$$\dot{y}_t = \alpha(x_t, y_t)\sigma^*(x_t, y_t)(\sigma\sigma^*)^{-1}(x_t, y_t)\big(\dot{x} - \overline{b}(x_t, y_t)\big) + \overline{\beta}(x_t, y_t)$$

with y_0 as in (26). Such a solution exists because of the requirements on σ. Now, setting

$$\phi_t = \sigma^*(x_t, y_t)(\sigma\sigma^*)^{-1}(x_t, y_t)\big(\dot{x} - \overline{b}(x_t, y_t)\big),$$

it immediately follows that $x_t = x_t(\phi)$ and $y_t = y_t(\phi)$, and the proof is concluded. $\square$

4.2 Diffusion processes satisfying a weak Hörmander condition: an example

In this section we treat an example of diffusion process which satisfies the weak Hörmander condition and has been recently studied in Bally and Kohatsu-Higa [4] (we are going to use the ideas and the estimates from that paper). Since lower bounds for the density have been already discussed in [4], we deal here only with the strict positivity. So, we give an application of our Theorem 7 in a case of degenerate diffusion coefficients.

We consider the diffusion process

$$X_t^1 = x^1 + \int_0^t \sigma_1(X_s)dW_s + \int_0^t b_1(X_s)ds, \quad X_t^2 = x^2 + \int_0^t b_2(X_s)ds \tag{31}$$

and we assume that $\sigma_1, b_1, b_2 \in C_b^\infty(\mathbb{R}^2; \mathbb{R})$. Actually, it suffices that they are four times differentiable - but we do not focus on this aspect here. Moreover, we fix some point $y \in \mathbb{R}^2$ and we assume that

$$|\sigma_1(y)| > c_* > 0 \quad \text{and} \quad |\partial_1 b_2(y)| > c_* > 0. \tag{32}$$

Let $\sigma = (\sigma_1, 0)^*$ and $b = (b_1, b_2)^*$. The Lie bracket $[\sigma, b]$ is computed as

$$\begin{aligned}
[\sigma, b](x) &= \partial_\sigma b(x) - \partial_b \sigma(x) \\
&= \begin{pmatrix} \sigma_1(x)\partial_1 b_1(x) - b_1(x)\partial_1\sigma_1(x) - b_2(x)\partial_2\sigma_1(x) \\ \sigma_1(x)\partial_1 b_2(x) \end{pmatrix}.
\end{aligned}$$

where for $f, g : \mathbb{R}^n \to \mathbb{R}^n$ smooth enough, we set $(\partial_f g)^i = f \cdot \nabla g^i$, $i = 1, \ldots, n$. So assumption (32) is equivalent with the fact that $\sigma(y)$ and $[\sigma, b](y)$ span $\mathbb{R}^2$, and this is the weak Hörmander condition in y.

We set $\overline{b} = b - \frac{1}{2}\partial_\sigma\sigma$ and for a measurable function $\phi \in L^2([0, T], \mathbb{R})$ we consider the skeleton $x(\phi)$, i.e. the solution of the equation

$$x_t(\phi) = x + \int_0^t \Big(\sigma(x_s(\phi))\phi_s + \overline{b}(x_s(\phi))\Big)ds.$$

Proposition 10. *Assume that $\sigma_1, b_1, b_2 \in C_b^\infty(\mathbb{R}^2)$ and (32) holds. Then the law of X_T has a local smooth density $p_T(x,\cdot)$ in a neighborhood of y. Moreover, if there exists a control $\phi \in L^2([0,T])$ such that $x_T(\phi) = y$ then $p_T(x,y) > 0$.*

Before starting with the proof of Proposition 10, let us consider the following decomposition: for $\delta \in (0,T]$, we set

$$F = X_T - x_T(\phi) \quad \text{ and } \quad F = F_{T-\delta} + G_\delta + R_\delta \tag{33}$$

where $F_{T-\delta} = X_{T-\delta} - x_{T-\delta}(\phi)$ and

$$\begin{aligned}
G_\delta^1 &= \int_{T-\delta}^T \sigma_1(X_{T-\delta})dW_s, \quad G_\delta^2 = \int_{T-\delta}^T \partial_\sigma b_2(X_{T-\delta})(T-s)dW_s \\
R_\delta^1 &= \int_{T-\delta}^T \Big(\sigma_1(X_s) - \sigma_1(X_{T-\delta})\Big)dW_s + \int_{T-\delta}^T b_1(X_s)ds + \\
&\quad - \int_{T-\delta}^T \Big(\sigma_1\big(x_s(\phi)\big)\phi_s + \overline{b}_1\big(x_{T-\delta}(\phi)\big)\Big)ds \\
R_\delta^2 &= \int_{T-\delta}^T \Big(\partial_\sigma b_2(X_s) - \partial_\sigma b_2(X_{T-\delta})\Big)(T-s)dW_s + \delta\Big(b_2(X_{T-\delta}) - b_2\big(x_{T-\delta}(\phi)\big)\Big) + \\
&\quad + \int_{T-\delta}^T Lb_2(X_s)(T-s)ds - \int_{T-\delta}^T \Big(b_2\big(x_s(\phi)\big) - b_2\big(x_{T-\delta}(\phi)\big)\Big)ds
\end{aligned}$$

in which $L = \frac{1}{2}\sigma\sigma^*\partial_x^2 + b\partial_x$ denotes the infinitesimal generator of X.
The covariance matrix of the conditional (on $\mathcal{F}_{T-\delta}$) Gaussian r.v. G_δ is given by

$$C_\delta = \delta\sigma_1^2(X_{T-\delta})\begin{pmatrix} 1 & \partial_1 b_2(X_{T-\delta})\frac{\delta}{2} \\ \partial_1 b_2(X_{T-\delta})\frac{\delta}{2} & (\partial_1 b_2)^2(X_{T-\delta})\frac{\delta^2}{3} \end{pmatrix}.$$

We need now some estimates which can be easily deduced from [4]. In order to be self contained, we propose here the following

Lemma 11. *Let $\rho_\delta^2 = \max(\delta, \int_{T-\delta}^T |\phi_s|^2 ds)$. Then, there exist $\delta_0 > 0$ such that for every $\delta < \delta_0$, on the set $\{|F_{T-\delta}| < \delta^{3/2}\rho_\delta\}$ the following properties hold:*

i) $\det C_\delta \geq c_1 \dfrac{\delta^4}{12}$;

ii) for every $\xi \in \mathbb{R}^2$, $|\xi|_\delta^2 \leq \dfrac{c_2}{\delta^3}\Big(\delta^2|\xi_1|^2 + |\xi_2|^2\Big)$; in particular, $|F_{T-\delta}|_\delta \leq c_2\rho_\delta$;

ii) for every $q \geq 2$, $\theta_{\delta,q} \leq L_q\rho_\delta$.

Here, c_1, c_2 and L_q are suitable positive constants depending on c_* and upper bounds for σ, b and their derivatives up to order 4, L_q depending on q also.

Proof. First, by recalling that σ and b are bounded, for some positive constant C we have

$$|x_{T-\delta}(\phi) - x_T(\phi)| \leq C\rho_\delta^2$$

so that

$$|X_{T-\delta} - y| \leq |X_{T-\delta} - x_{T-\delta}(\phi)| + C\rho_\delta^2.$$

Therefore, we can choose δ_0 such that for all $\delta < \delta_0$ the following holds: if $|F_{T-\delta}| = |X_{T-\delta} - x_{T-\delta}(\phi)| < \delta^{3/2}\rho_\delta$ then

$$|\sigma_1(X_{T-\delta})| \geq c_* > 0 \quad \text{and} \quad |\partial_1 b_2(X_{T-\delta})| \geq c_* > 0.$$

Therefore,

$$\det C_\delta = \frac{(\sigma_1^2 \partial_1 b_2)^2(X_{T-\delta})\delta^4}{12} \geq c_1\delta^4$$

and $i)$ holds. Moreover, we have

$$C_\delta^{-1} = \frac{1}{(\sigma_1\partial_1 b_2)^2(X_{T-\delta})\delta^3}\begin{pmatrix} 4(\partial_1 b_2)^2(X_{T-\delta})\delta^2 & -6(\partial_1 b_2)(X_{T-\delta})\delta \\ -6(\partial_1 b_2)(X_{T-\delta})\delta & 12 \end{pmatrix}$$

so that for $\xi \in \mathbb{R}^2$,

$$\begin{aligned}
|C_\delta^{-1/2}\xi|^2 = \langle C_\delta^{-1}\xi, \xi\rangle &= \frac{1}{(\sigma_1\partial_1 b_2)^2(X_{T-\delta})\delta^3}\Big(\big(2\partial_1 b_2(X_{T-\delta})\delta\xi_1 - 3\xi_2\big)^2 + 3\xi_2^2\Big) \\
&\leq \frac{C}{c_*^4\delta^3}\Big(\delta^2|\xi_1|^2 + |\xi_2|^2\Big)
\end{aligned}$$

where C depends on σ and b. Then, if $|F_{T-\delta}| < \delta^{3/2}\rho_\delta$ one gets

$$|F_{T-\delta}|_\delta^2 = |C_\delta^{-1/2}F_{T-\delta}|^2 \leq \frac{C}{c_*^4\delta^3}\delta^3\rho_\delta^2(\delta^2+1) \leq c_2\rho_\delta^2$$

and $ii)$ is proved. As for $iii)$, for $q \geq 2$ we have

$$\mathbb{E}\big(|C_\delta^{-1/2}R_\delta|^q\big) \leq \Lambda_q\Big(\mathbb{E}(|\delta^{-1/2}R_\delta^1|^q) + \mathbb{E}(|\delta^{-3/2}R_\delta^2|^q)\Big),$$

where Λ_q depends on q, c_*, σ and b. Now, by using the Burkholder inequality and the boundedness of the coefficients b and σ and of their derivatives, one has

$$\begin{aligned}
\mathbb{E}(|\delta^{-1/2}R_\delta^1|^q) \leq C_q\delta^{-q/2}\Big[&\mathbb{E}\Big(\Big|\int_{T-\delta}^T \big(\sigma_1(X_s) - \sigma_1(X_{T-\delta})\big)dW_s\Big|^q\Big) + \\
&+ \mathbb{E}\Big(\Big|\int_{T-\delta}^T b_1(X_s)ds\Big|^q\Big) + \\
&+ \mathbb{E}\Big(\Big|\int_{T-\delta}^T \Big(\sigma_1\big(x_s(\phi)\big)\phi_s + \overline{b}_1\big(x_{T-\delta}(\phi)\big)\Big)ds\Big|^q\Big)\Big] \\
\leq C_q C\delta^{-q/2}\cdot\Big(\delta^q + \delta^{q/2}&\Big(\int_{T-\delta}^T |\phi_s|^2 ds\Big)^{q/2}\Big) \leq 2C_q C\rho_\delta^q
\end{aligned}$$

where C_q depends on q only and C depends on the bounds of the diffusion coefficients. Similarly (in the following C denotes a suitable constant),

$$\begin{aligned}
\mathbb{E}(|\delta^{-3/2}R^2_\delta|^q) &\leq C_q\delta^{-3q/2}\Big[\mathbb{E}\Big(\Big|\int_{T-\delta}^T\big(\partial_\sigma b_2(X_s)-\partial_\sigma b_2(X_{T-\delta})\big)(T-s)dW_s\Big|^q\Big)+\\
&\qquad+\mathbb{E}\Big(\delta^q\big|b_2(X_{T-\delta})-b_2\big(x_{T-\delta}(\phi)\big)\big|^q\Big)+\\
&\qquad+\mathbb{E}\Big(\Big|\int_{T-\delta}^T Lb_2(X_s)(T-s)ds\Big|^q\Big)+\\
&\qquad+\Big|\int_{T-\delta}^T\Big(b_2\big(x_s(\phi)\big)-b_2\big(x_{T-\delta}(\phi)\big)\Big)ds\Big|^q\Big]\\
&\leq 2C_qC\delta^{-3q/2}\Big(\delta^{2q}+\delta^q|F_{T-\delta}|^q+\delta^q\sup_{T-\delta\leq s\leq T}|x_s(\phi)-x_{T-\delta}(\phi)|^q\Big)\\
&\leq 2C_qC\delta^{-3q/2}\Big(\delta^{2q}+\delta^q\cdot\delta^{3q/2}\rho^q_\delta+\delta^q\cdot\Big[\delta^q+\delta^{q/2}\cdot\Big(\int_{T-\delta}^T|\phi_s|^2ds\Big)^{q/2}\Big]\Big)\\
&\leq C_qC\rho^q_\delta.
\end{aligned}$$

The same arguments may be used to give upper estimates for the remaining terms in $\|C_\delta^{-1/2}R_\delta\|^q_{\delta,2,q}$ that contain the Malliavin derivatives. So, we deduce that

$$\|C_\delta^{-1/2}R_\delta\|_{\delta,2,q}\leq L_q\rho_\delta$$

and the proof is completed. $\square$

We are now ready for the

Proof of Proposition 10. Consider the decomposition (33): we have $p_{X_T}(y)=p_F(0)$. We use Lemma 11 and Theorem 7 (notations come from these results). So, there exists δ_0 such that for $\delta<\delta_0$ if $|F_{T-\delta}|<\delta^{3/2}\rho_\delta$ then $|F_{T-\delta}|_\delta<c_2\rho_\delta$. We take now $\delta_1<\delta_0$ and $r=c_2\rho_{\delta_1}$. So, there exists $\delta<\delta_1$ such that $\theta_{\delta,q_d}=\|C_\delta^{-1/2}R_\delta\|_{\delta,2,q_d}\leq a(r)$. Therefore, $\Gamma_{\delta,r}(0)\supset\{|F_{T-\delta}|<\delta^{3/2}\rho_\delta\}$ and by using Theorem 7 we get

$$\begin{aligned}
p_{X_T}(y)=p_F(0)&\geq\frac{1}{2e^{r^2}(2\pi)^{d/2}}\,\mathbb{E}\big((\det C_\delta)^{-1/2}\mathbf{1}_{\{|F_{T-\delta}|<\delta^{3/2}\rho_\delta\}}\big)\\
&\geq\frac{1}{2e^{r^2}(2\pi)^{d/2}}\,c_1\frac{\delta^4}{12}\,\mathbb{P}(|F_{T-\delta}|<\delta^{3/2}\rho_\delta).
\end{aligned}$$

Finally, $F_{T-\delta}=X_{T-\delta}-x_{T-\delta}(\phi)$: by the support theorem, $\mathbb{P}(|X_{T-\delta}-x_{T-\delta}(\phi)|<\varepsilon)>0$ for all $\varepsilon>0$, so that $p_{X_T}(y)>0$. $\square$

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